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On–off diffusion: onset and statistics

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Abstract. A particle motion in a 2D periodic potential with the symmetry such that a particular motion can be restricted on the x-axis, subject to the external periodic force in the x-direction, is studied. It is found that by changing the amplitude of the external force, the 1D diffusive motion in the x-direction undergoes the instability at an amplitude, above which the diffusive motion in the y-direction, showing on–off intermittency, is observed. We call it *on–off diffusion*. By introducing a simple mapping model, the diffusion coefficient in the y-direction is found to take the scaling form $D_\perp = \lambda_\perp^p h(\lambda_\perp^{-q} L)$ slightly above the instability point where $p \simeq 1.42$ and $q \simeq 1.95$. $\lambda_\perp (> 0)$ and L, respectively, are the transverse Lyapunov exponent evaluating the magnitude of instability and its fluctuation in the y-direction. The scaling function $h(z)$ takes the asymptotic form, $h(z) = \text{const.}$ for $0 < z \ll 1$ and $\propto z^\alpha$ ($\alpha \simeq 0.38$) for $z \gg 1$.

1. Introduction

Diffusion is not specific to systems under the influence of external random forces such as the Brownian motion but is also observed in chaotic dynamical systems, provided that the systems have appropriate spatial translational symmetry. The diffusion constant is determined by the autocorrelation function of the particle velocity. The mixing property, which is one of the important characteristics of chaos, can ensure a non-vanishing diffusion coefficient in chaotic systems. The diffusion in chaotic systems is called the *deterministic diffusion* or the *chaos-induced diffusion* [1–5], and many studies have so far been reported.

Several years ago, Geisel *et al* reported the possibility of the existence of the transition between 1D and 2D diffusions in Hamiltonian systems. This is quite an interesting phenomenon in transport theory [6, 7].

On the other hand, many studies have recently been reported on on–off intermittency which is a phenomenon typically observed when a particular chaotic motion in a coupled oscillator system in a wide sense undergoes the instability [8]. On–off intermittency has been observed not only in numerical models [9–11] but also in laboratory experiments [12–16]. Recently, Ott and Sommerer proposed a new dissipative dynamical model showing on–off intermittency [17]. In their model, on–off intermittency is observed when a 1D particular motion loses its stability and 2D motion sets in.

Furthermore, Lai and Grebogi [18] extended the Ott–Sommerer model in such a way that two equivalent subspaces showed physically equivalent 1D particular motion. Namely, this model has two symmetric invariant subspaces. As the parameter of a system changes, the largest Lyapunov exponent transverse to the invariant subspaces changes its sign. When the transverse Lyapunov exponent is positive, the state point in the vicinity of one of the invariant subspaces jumps into the other invariant subspace, exhibiting on–off intermittency. This is called *two-state on–off intermittency*.

In this paper we give a new bifurcation phenomenon leading to the 2D diffusive motion as a result of the instability of 1D diffusive motion, exhibiting on–off intermittent characterization. This paper is constructed as follows. In section 2, proposing a new dynamical model, we report a new bifurcation showing on–off intermittency associated with the transition between 1D and 2D diffusions. This will be called *on–off diffusion*. In section 3, we introduce a mapping model in order to discuss the critical statistics of on–off diffusion and we will find a new scaling law just after the onset of on–off diffusion. The summary and concluding remarks are given in section 4 and a short review of on–off intermittency is added in the appendix.

2. On–off diffusion

2.1. Transition between 1D and 2D diffusions

We consider a dissipative dynamical system where a particle moves under the influence of a 2D potential $U(\boldsymbol{x})$ and is subject to a periodic forcing in the x-direction. The equation of motion is written as

$$\ddot{\boldsymbol{x}}(t) = -\gamma\dot{\boldsymbol{x}}(t) - \nabla U(\boldsymbol{x}(t)) + f\sin(\Omega t)\boldsymbol{e}_x \tag{1}$$

where $\boldsymbol{x}(t) = (x(t), y(t))$ denotes the position of the particle at time t and γ is the friction coefficient. f and Ω are, respectively, the amplitude and the frequency of the periodic external force applied in the x-direction with $\boldsymbol{e}_x$ being the corresponding unit vector. We assume that the potential is a periodic function of period 1 in both the x- and y-directions (i.e. $U(x+1, y) = U(x, y+1) = U(x, y)$) and satisfies

$$\left.\frac{\partial U(x, y)}{\partial y}\right|_{y=N} = 0 \qquad N = 0, \pm 1, \pm 2, \ldots \tag{2}$$

for an arbitrary x-value.

Due to the translational symmetry of the potential and the property (2), the equation of motion admits a particular solution $\boldsymbol{x} = \boldsymbol{x}_0 = (x_0(t), N)$ where $x_0(t)$ obeys

$$\ddot{x}_0 = -\gamma\dot{x}_0 - \frac{\partial U(x_0, N)}{\partial x_0} + f\sin(\Omega t) \tag{3}$$

and N is an arbitrary constant integer. This solution is equal to the motion on the 3D invariant manifold in the 5D phase space spanned by $(x, \dot{x}, y, \dot{y}, t)$. It displays a 1D motion restricted on the x-axis in the real space. To examine the trajectory instability, the linear stability analysis of (3) will be performed by putting

$$x(t) = x_0(t) + \delta x(t) \qquad \dot{x}(t) = \dot{x}_0(t) + \delta\dot{x}(t). \tag{4}$$

Let us define the exponent

$$\lambda_\parallel = \lim_{t\to\infty}\frac{1}{t}\log\frac{\rho_\parallel(t)}{\rho_\parallel(0)} \tag{5}$$

where $\rho_\parallel(t) \equiv (\delta x(t)^2 + \delta\dot{x}(t)^2)^{1/2}$. The exponent $\lambda_\parallel$ is identical to the *largest Lyapunov exponent* of the 1D motion (the particular solution) in the x-direction, being negative and positive, respectively, for periodic and chaotic motions.

We hereafter consider the case $\lambda_\parallel > 0$, where the particle in a particular motion displays a chaotic motion. If the particle jumps over potential valleys, the 1D diffusion can exist since $U(x, N)$ is periodic with respect to x. The diffusion coefficient $D_\parallel$ is given by

$$\sigma_\parallel^2(t) \equiv \langle (x_0(s+t) - x_0(s))^2 \rangle_s = 2D_\parallel t \tag{6}$$

for $t \to \infty$, where $\langle h(s) \rangle_s$ denotes the long time average $\lim_{T\to\infty} T^{-1} \int_0^T h(s)\, \mathrm{d}s$. In this paper, particular attention is paid when no drift motion is present.

The infinitesimal perturbation $(y, \dot{y})$ transverse to the invariant manifold is examined with

$$\ddot{y} = -\gamma \dot{y} - \left. \frac{\partial^2 U(x_0(t), y)}{\partial y^2} \right|_{y=0} y \tag{7}$$

where $x_0(t)$ is obtained by (3).

The stability of the variation in the transverse direction to the invariant manifold is examined with the distance from the particular motion $\rho_\perp(t) \equiv (y(t)^2 + \dot{y}(t)^2)^{1/2}$. The *transverse Lyapunov exponent* $\lambda_\perp$ defined by

$$\lambda_\perp = \lim_{t\to\infty} \frac{1}{t} \log \frac{\rho_\perp(t)}{\rho_\perp(0)} \tag{8}$$

displays the growth rate of the distance from the particular motion in the transverse direction. If $\lambda_\perp > 0$, the 1D motion is unstable and a spatially 2D motion is eventually observed. The local transverse expansion rate introduced by $\Lambda_\perp(t) \equiv \dot{\rho}_\perp(t)/\rho_\perp(t)$, displays a fluctuation around the mean value $\lambda_\perp$. The fluctuation $R_\perp(t) \equiv \Lambda_\perp(t) - \lambda_\perp$ is caused by the chaotic motion on the invariant manifold corresponding to the 1D motion and its intensity is given by

$$L = \int_0^\infty \langle R_\perp(t+s) R_\perp(s) \rangle_s \, \mathrm{d}t. \tag{9}$$

Since the potential is periodic in the y-direction, we expect that the instability in the y-direction leads to a large-scale motion associated with the diffusion. The diffusion coefficient $D_\perp$ in the y-direction is defined via

$$\sigma_\perp^2(t) \equiv \langle (y(t+s) - y(s))^2 \rangle_s = 2 D_\perp t \tag{10}$$

for $t \to \infty$. Therefore, the diffusion coefficient is given by

$$D_\perp = \int_0^\infty \langle \dot{y}(t+s) \dot{y}(s) \rangle_s \, \mathrm{d}t. \tag{11}$$

Needless to say, when the 1D motion is stable, $D_\perp$ vanishes.

2.2. Onset of on–off diffusion

As a concrete model, we take the potential

$$U(x, y) = -\frac{a}{\pi} \cos^2(\pi x) \cos^2(\pi y) \qquad a > 0. \tag{12}$$

This potential satisfies all the properties given above (figure 1).

In the numerical simulation, parameters are fixed as $\gamma = 0.06$, $a = 1.6$, $\Omega = 1.0$ and the value of the amplitude f is changed. We find the transition points $f_{c1} \sim 1.7$ and $f_{c2} \sim 4.0$. For $f = 4.1$, we observe the 1D diffusive motion. For $f = 3.7$, we find $\lambda_\parallel > 0$ and $\lambda_\perp$ is slightly positive which implies the particular motion is unstable, and therefore the transversal motion occurs (figure 2). The diffusion coefficient in the y-direction is quite small compared with that in the x-direction (figure 3). The velocity $\dot{y}$ turns out to display an intermittency. We calculate the probability distribution $P(|\dot{y}|)$, the power spectrum $I(\omega)$ and the laminar duration distribution $Q(\tau)$ for the intermittent variable $\dot{y}$. Numerical results show the power law form $P(|\dot{y}|) \sim |\dot{y}|^{-1+\xi}$, $(|\dot{y}| \ll 1)$ where ξ is small, $I(\omega) \sim \omega^{-1/2}$ in the low-frequency region, and $Q(\tau) \sim \tau^{-3/2}$ in an intermediate region of τ as shown in figure 4. These results are in agreement with the well known statistics of on–off intermittency (appendix). For that reason, this diffusion in the y-direction generated via on–off intermittency is called '*on–off diffusion*'.

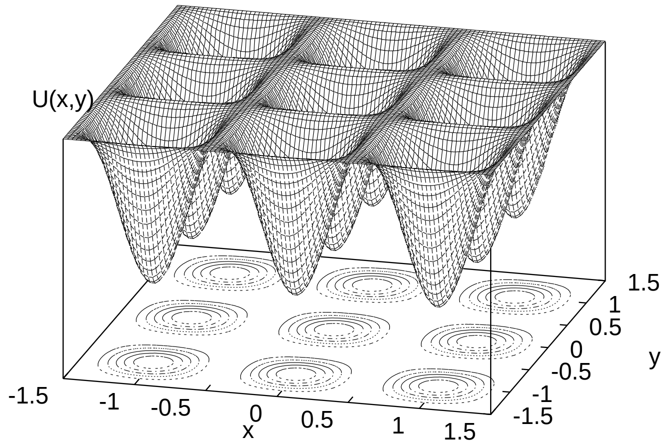

Figure 1. The potential $U(x, y)$ has spatial translational symmetry in the x- and y-directions.

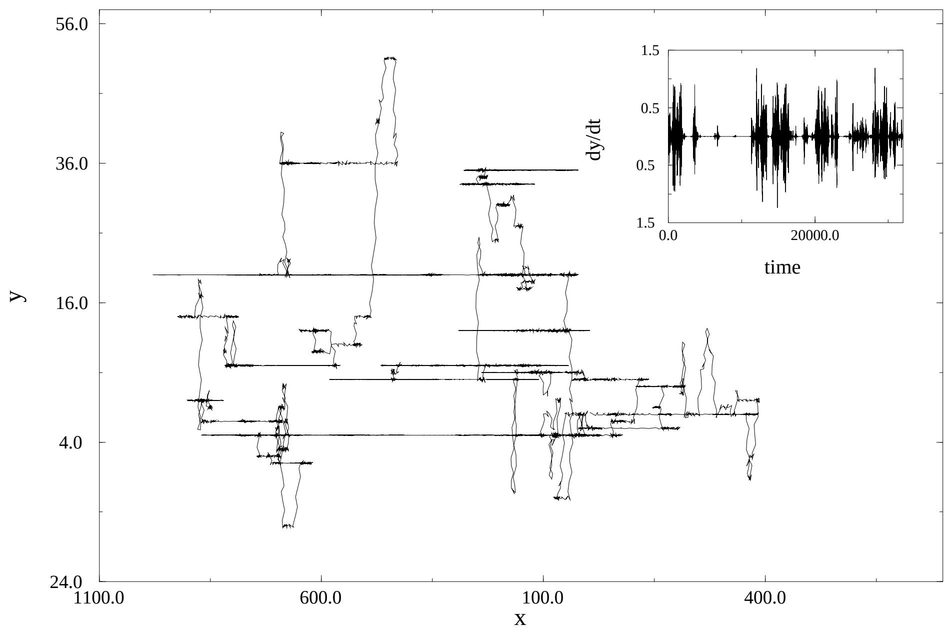

Figure 2. On–off diffusion on the x–y plane with the parameter value $f = 3.7$. Time series of $\dot{y}$ plotted in the enlargement displays on–off intermittency.

3. A discrete mapping model of on–off diffusion and statistical analysis

Since the differential equation system (1) has many windows in the parameter space the critical region associated with the onset of on–off diffusion is quite narrow. Therefore, it is difficult to analyse statistical properties near the transition point separating regions showing 1D and 2D diffusions. In order to study the critical dynamics, we propose a more simplified model. Focusing on the motion in the y-direction which displays on–off diffusion, we introduce a 1D discrete mapping system that keeps the property of on–off diffusion.

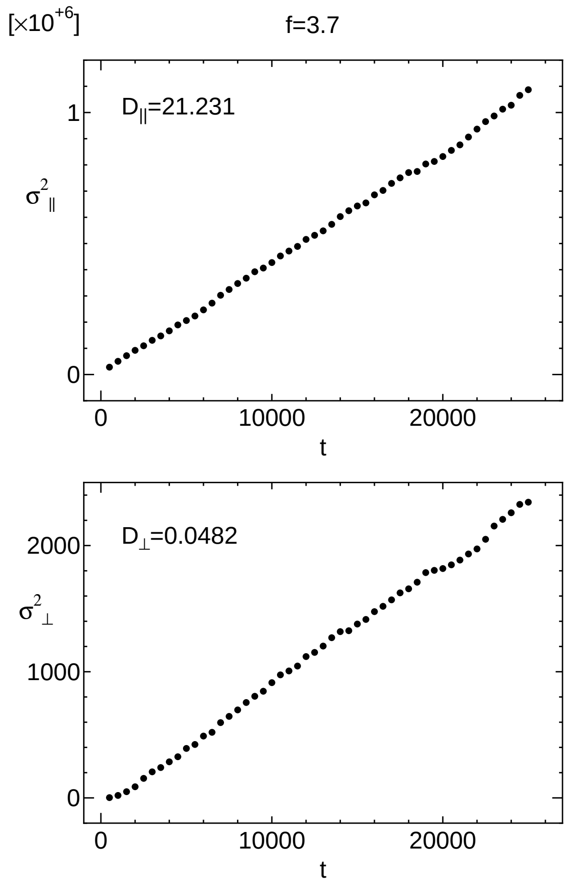

Figure 3. Variance in the x- and y-direction for $f = 3.7$ with the diffusion coefficient as, respectively, $D_{\|} = 21.231$ and $D_{\perp} = 0.0482$.

The particle position in the y-direction at time step $n = 0, 1, 2, \ldots$ is denoted by Y_n which is divided into the cell number N_n and the position η_n in the cell, ($|\eta_n| < \frac{1}{2}$) i.e., $Y_n = N_n + \eta_n$. Noting that the statistical property of on–off intermittency is similar to that of the type III intermittency [19], we employ the discrete system

$$Y_{n+1} = F(Y_n, \Lambda_n) = N_n + f(\eta_n, \Lambda_n) \qquad f(\eta, \Lambda) \equiv \mathrm{e}^{\Lambda}\eta + 4\eta^3. \tag{13}$$

The mapping function $F(Y, \Lambda)$ is a periodic function, $F(Y + 1, \Lambda) = F(Y, \Lambda)$ and an antisymmetry relation $f(-\eta, \Lambda) = -f(\eta, \Lambda)$ is assumed. The last property reflects the inversion symmetry of the potential in the y-direction (figure 5). The stability of fixed points ($\eta = 0$) is determined by the transverse Lyapunov exponent $\lambda_{\perp}$ which is the average of the local transverse expansion rate Λ_n. Λ_n is decomposed into $\Lambda_n = \lambda_{\perp} + R_n$, where R_n is the fluctuation correlated by the chaotic motion in the x-direction. The inequality $\Lambda_n < 0$ ($\Lambda_n > 0$) means the fixed point is momentarily stable (unstable). For simplicity, we assume the temporal fluctuation R_n is a Gaussian white noise with the statistics $\langle R_n \rangle = 0$ and $\langle R_n R_{n'} \rangle = 2L\delta_{nn'}$.

The fluctuation of Λ_n may cause an intermittency in the cell. If $\lambda_{\perp} < 0$, the particle reaches one of the fixed points $\eta = 0$, $(Y = N)$ via a transient intermittency. This means the particle eventually displays a chaotic motion only in the x-direction. If $\lambda_{\perp} > 0$, fixed points

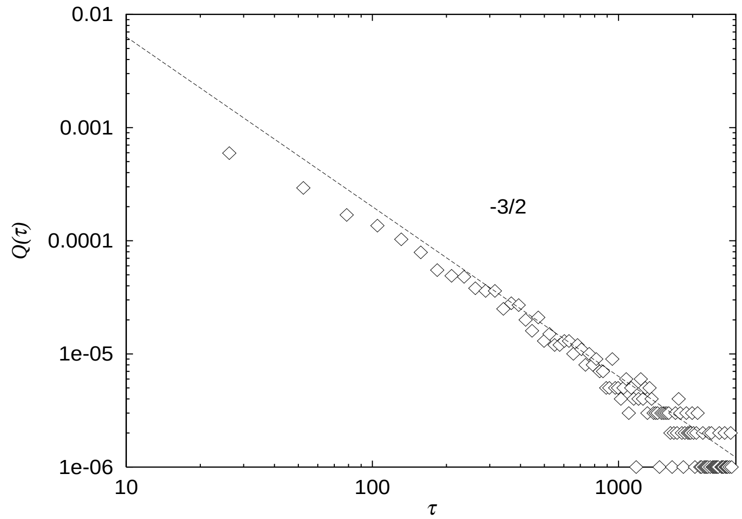

Figure 4. Laminar duration distribution of $\dot{y}$ for $f = 3.7$. The broken line indicates the theoretical result $Q(\tau) \sim \tau^{-3/2}$.

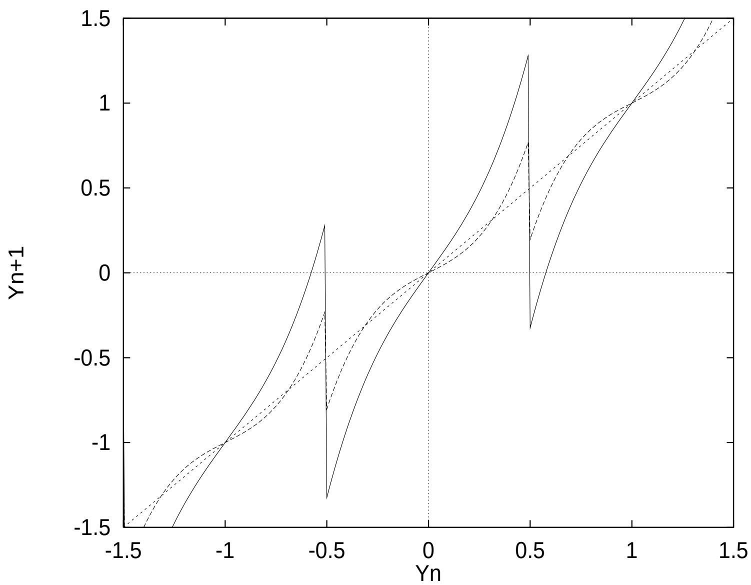

Figure 5. The mapping function $F(\eta_n, \Lambda_n)$ for $\Lambda_n = -0.5$ (broken curve) and 0.5 (full curve). Fixed points $Y_n = 0, \pm 1, \pm 2 \ldots$ are momentarily stable (unstable) with $\Lambda_n < 0$ ($\Lambda_n > 0$).

are unstable, which corresponds to the onset of the 2D motion in the model in the preceding section. This fact implies that the particle moves from cell to cell in an irregular way (the onset of diffusion).

Equation (13) can be uniquely decomposed into the following two dynamics;

$$N_{n+1} = N_n + J(\eta_n, \Lambda_n) \tag{14}$$

$$\eta_{n+1} = g(\eta_n, \Lambda_n) \tag{15}$$

where $J(\eta_n, \Lambda_n) \equiv [f(\eta_n, \Lambda_n) \pm \frac{1}{2}]$ $(Y_n \gtrless 0)$, $[Z]$ being the nearest integer of Z, which is equal to the jumping number in a unit time step and $g(\eta_n, \Lambda_n) = f(\eta_n, \Lambda_n) - J(\eta_n, \Lambda_n)$. Now we define the probability distribution for η_n,

$$P_n(\eta) \equiv \langle\langle \delta(\eta_n - \eta) \rangle\rangle \tag{16}$$

where $\langle\langle \cdots \rangle\rangle$ is the average over both the initial value η_0 and the fluctuations $R_0, R_1, \ldots, R_{n-1}$.

It is easy to observe that the probability distribution obeys the Frobenius–Perron equation

$$\begin{aligned} P_{n+1}(\eta) &= \int_{-\infty}^{\infty} \mathrm{d}R \frac{\mathrm{e}^{-R^2/4L}}{\sqrt{4\pi L}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \mathrm{d}\zeta \; \delta(g(\zeta, \Lambda) - \eta) P_n(\zeta) \\ &= \sum_j \int_{-\infty}^{\infty} \mathrm{d}R \frac{\mathrm{e}^{-R^2/4L}}{\sqrt{4\pi L}} \frac{P_n(\tilde{\eta}_j)}{|g'(\tilde{\eta}_j, \Lambda)|} \equiv H(\eta) P_n(\eta) \end{aligned} \tag{17}$$

where $g'(\eta, \Lambda) \equiv \partial g(\eta, \Lambda)/\partial\eta$, and the summation is taken over all $\tilde{\eta}_j$'s which satisfy $g(\tilde{\eta}_j, \Lambda) = \eta$. A slight calculation shows that the steady state distribution $P_*(\eta)(= H(\eta)P_*(\eta))$ satisfies

$$P_*(\eta) = C|\eta|^{-1+\lambda_\perp/L} \tag{18}$$

for small η, where C is a constant.

Numerical calculation is carried out by setting $\lambda_\perp = 10^{-4}$ and $L = 10^{-2}$. We observed that the probability distribution for the distance η takes a power law $P_*(\eta) \sim |\eta|^{-1+\xi}$ and that ξ is well approximated by $\xi = \lambda_\perp/L$ for small η. By numerically solving (14) and (15), it is also found that the power spectrum takes $I(\omega) \sim \omega^{-1/2}$ in a low-frequency region, and that the laminar duration distribution $Q(\tau)$ takes $Q(\tau) \sim \tau^{-3/2}$. These statistical features are in agreement with those of on–off intermittency.

Numerically solving (14) and (15), we calculated the dispersion

$$\begin{aligned} \sigma_n^2 &= \langle\langle (N_n - N_0)^2 \rangle\rangle \\ &= \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \langle\langle J(\eta_i, R_i) J(\eta_j, R_j) \rangle\rangle. \end{aligned} \tag{19}$$

We obtained that it asymptotically takes $\sigma_n^2 \simeq 2D_\perp n$ for large n. For small $\lambda_\perp$ and L that satisfies $\lambda_\perp \leqslant L$, the transverse diffusion coefficient $D_\perp$ turns out to have the scaling form

$$D_\perp(\lambda_\perp, L) = \lambda_\perp^p h(\lambda_\perp^{-q} L) \tag{20}$$

where $p \simeq 1.42$, $q \simeq 1.95$. $h(z)$ is the scaling function with the asymptotic forms

$$h(z) \sim \begin{cases} \text{const.} & (0 < z \ll 1) \\ z^\alpha & (z \gg 1) \end{cases} \tag{21}$$

where $\alpha \simeq 0.38$ (figure 6).

4. Summary and concluding remarks

In this study, we extended the Lai–Grebogi model for two-state on–off intermittency in such a way that the system exhibits an *infinite-state on–off intermittency* by utilizing the periodic potential. The periodic property of the potential produces the possibility of the diffusive behaviours of the particle.

We studied the transition between 1D and 2D diffusions by changing the intensity of the external force. For $\lambda_\parallel > 0$ and $\lambda_\perp < 0$, a 1D diffusion in the x-direction is observed. For

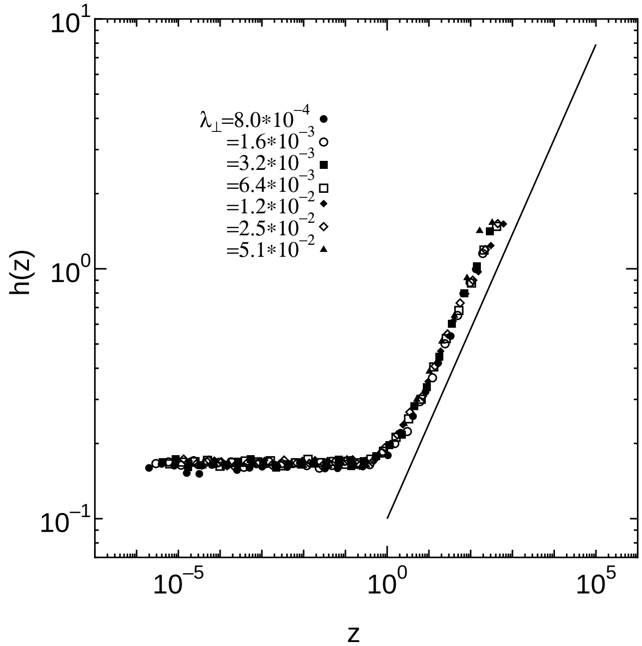

Figure 6. Scaling function $h(z)$ ($z = \lambda_\perp^{-1.95} L$) is plotted with various parameter values of $\lambda_\perp$.

slightly positive $\lambda_\perp$, the particular motion becomes weakly unstable, and a diffusion triggered by on–off intermittency in a direction perpendicular to the external force is observed. This paper is the first report of such a new bifurcation phenomenon and we called the diffusive motion associated with on–off intermittency *on–off diffusion*. When a strong burst occurs near one of the invariant manifolds ($x = x_0(t)$, $y = 0, \pm 1, \pm 2, \ldots;\ \dot{y} = 0$) with the sufficient energy for jump over the potential valley, the particle moves onto another invariant manifold. This process is repeated in an irregular manner, and the particle starts to itinerate among an infinite number of invariant manifolds. This is the origin of on–off diffusion.

For $\lambda_\parallel > 0$ and $\lambda_\perp < 0$, the present system has infinitely many attractors where a particle shows 1D diffusion restricted in the x-direction. In the Lai–Grebogi model for parameter values where two-state on–off intermittency is not observed, there are points which belong to the basin of one attractor near any point in the basin of the other attractor. Such a complex basin structure is called *intermingled*. In this connection it is expected that the present system also have the intermingled basin structures for $\lambda_\parallel > 0$ and $\lambda_\perp < 0$.

The present potential is periodic not only in the x- and y-directions, but also in the $x'(\equiv x + y)$- and $y'(\equiv x - y)$-directions. If the external force is applied in the x'-direction, changing its intensity, we may expect the observation of on–off diffusion discussed above. We carried out numerical integration in such a case for various parameter values. However, when a 1D periodic motion in the x'-direction loses its stability, we always observed a 2D diffusive motion both in the x'- and y'-directions. Namely, the instability of the 1D periodic motion always causes a simultaneous onset of diffusions in both x'- and y'-directions and we have not observed a 1D diffusive motion in the x'-direction. This fact is quite different from the situation studied in this paper. This may be related to the fact that the maximum regions of the potential where the particle showing 1D motion passes through are quite different from each other in the two cases.

Furthermore, to estimate the dependence of the diffusion coefficient $D_\perp$ on both $\lambda_\perp$ and L after the instability point, we introduced a simple mapping model exhibiting the same statistical features as on–off intermittency. We analysed the details of the statistics of the diffusion after the onset of on–off diffusion, and found a scaling law for the diffusion coefficient with respect to the magnitude of the instability $\lambda_\perp$ and the fluctuation intensity of the local transverse expansion rate.

In this paper, we used a potential which is periodic in both the x- and y-directions and on–off intermittency causes in the y-direction. Therefore, the transition separates the 1D and 2D diffusive behaviours. However, if the potential $U(x, y)$ has a form such that the particle is always located in a certain finite region in the x-direction and is periodic only in the y-direction, on–off diffusion in the y-direction can be also observed as far as the 1D chaotic motion in the x-direction undergoes the instability. In this sense, the periodic property of the potential in the x-direction is not necessary for the observation of on–off diffusion in the y-direction.

Finally, it should be noted that on–off diffusion is observed not in a specific model, but is quite a universal phenomenon as far as the potential has a certain kind of symmetry. We expect that the on–off diffusion can be observed in various systems both experimentally in real experiments and numerically in other mathematical models.

Appendix. On–off intermittency (Intermittency caused by chaotic modulation)

Let us consider a complex dynamical system, where the state variables $\boldsymbol{X}$ and $\boldsymbol{v}$ obey

$$\dot{\boldsymbol{X}}(t) = \boldsymbol{F}(\boldsymbol{X}, \boldsymbol{v}) \qquad \dot{\boldsymbol{v}}(t) = \boldsymbol{G}(\boldsymbol{X}, \boldsymbol{v}). \tag{A.1}$$

If $\boldsymbol{G}$ has an antisymmetric property $\boldsymbol{G}(\boldsymbol{X}, -\boldsymbol{v}) = -\boldsymbol{G}(\boldsymbol{X}, \boldsymbol{v})$, equation (A.1) has a particular motion

$$\dot{\boldsymbol{X}}^0(t) = \boldsymbol{F}(\boldsymbol{X}^0, \mathbf{o}) \qquad \boldsymbol{v}^0(t) = \mathbf{o}. \tag{A.2}$$

Then the subspace corresponding to the motion is called the *invariant manifold*.

We examine the stability of the motion on the invariant manifold occupied by the motion given by equation (A.2) with the two exponents

$$\lambda_\parallel \equiv \lim_{t\to\infty} \frac{1}{t} \log \frac{|\delta \boldsymbol{X}(t)|}{|\delta \boldsymbol{X}(0)|} \qquad \lambda_\perp \equiv \lim_{t\to\infty} \frac{1}{t} \log \frac{|\delta \boldsymbol{v}(t)|}{|\delta \boldsymbol{v}(0)|}. \tag{A.3}$$

They are given by two linearized equations for $\delta\boldsymbol{X}(t)$ and $\delta\boldsymbol{v}(t)$, where $\boldsymbol{X}(t) = \boldsymbol{X}^0(t) + \delta\boldsymbol{X}(t)$, $\boldsymbol{v}(t) = \delta\boldsymbol{v}(t)$. The exponent $\lambda_\parallel$ is identical to the largest Lyapunov exponent of the particular motion (A.2), being negative for periodic motion and positive for chaotic motion. The exponent $\lambda_\perp$ called the *transverse Lyapunov exponent* is relevant to the stability of the particular motion (equation (A.2)) with respect to perturbation vector transverse to the manifold. Now we assume that $\lambda_\perp$ changes its sign from negative to positive values, the particular motion keeping chaotic motion ($\lambda_\parallel > 0$). If the local expansion rate has a strong fluctuation, the chaotic orbit spends long stretches of time near the invariant manifold and every once in a while, it experiences a burst, in which it moves far from the invariant manifold. The repeat of this process shows highly intermittent behaviour. This is called *on–off intermittency*.

Statistical features of on–off intermittency are summarized as follows. The probability density for the intermittency variable $\rho(t)$ ($\equiv |\boldsymbol{v}(t)|$, the distance transverse to the invariant manifold) takes a power law form $P_*(\rho) \sim \rho^{-1+\xi}$ for small ξ [20], where the exponent ξ is proportional to the transverse Lyapunov exponent. Furthermore the power spectrum takes a power law form $I(\omega) \sim \omega^{-1/2}$ in a low-frequency region [20,21], and the distribution function for the duration of a laminar motion $Q(\tau)$ takes a power law dependence $Q(\tau) \sim \tau^{-3/2}$

in an intermediate region of τ [22]. These statistical features are caused by self-similar characteristics of on–off intermittency.

These statistical features can be described by a simple stochastic model [20]. A time evolution of $\rho(t)$ is assumed to be given, by taking into account the nonlinear effect, as

$$\dot{\rho}(t) = (\lambda_{\perp} + R(t))\rho(t) - \beta\rho(t)^3 \tag{A.4}$$

where β is positive constant. This is identical to the so-called multiplicative noise model. The local transverse expansion rate is replaced by $\Lambda_{\perp}(t) \equiv \lambda_{\perp} + R(t)$, the fluctuation $R(t)$ being assumed to be a Gaussian white noise with $\langle R(t)\rangle = 0$, $\langle R(t)R(0)\rangle = 2L\delta(t)$. Then the Fokker–Plank equation is written as

$$\frac{\partial P(\rho,t)}{\partial t} = L\frac{\partial}{\partial\rho}\left[\rho^2 P_*(\rho)\frac{\partial}{\partial\rho}\left[\frac{P(\rho,t)}{P_*(\rho)}\right]\right] \tag{A.5}$$

where $P_*(\rho) \sim \rho^{-1+\xi}\mathrm{e}^{-\beta\rho^2/2L}$ is the steady-state distribution. It brings a power law $P_*(\rho) \sim \rho^{-1+\xi}$, $\xi = \lambda_{\perp}/L$ for small ρ [20]. We can derive from equation (A.5) the asymptotic form of the power spectrum $I(\omega) \sim \omega^{-1/2}$ in the low-frequency region [20, 21].

Recently, Miyazaki and Hata proposed a solvable mapping model of on–off intermittency [23, 24]. They rigorously found the above asymptotic forms with the solvable model.

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